

Pairs of k -free Numbers, consecutive square-full Numbers

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Abstract

We consider the error term of the asymptotic formula for the number of pairs of k -free integers up to x . Our error term improves previous results by Heath-Brown and Brandes for $k \leq 17$. We then extend our results to r -tuples of k -free numbers and improve previous results by Tsang. Finally, we establish an error term for consecutive square-full integers. The main tool of our work is the approximate Determinant Method.

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1 Introduction

An integer n is called square-free if there is no prime p such that $p^2 \mid n$. More generally, for $k \in \mathbb{Z}_{\geq 2}$, we say that an integer n is k -free if there is no prime p such that $p^k \mid n$. One can observe that k -freeness is a fairly desirable property for integers. For example, one can take difficult statements about prime numbers and consider the equivalent statements about k -free numbers. For instance, while the twin prime conjecture is a very hard problem, we can answer the problem of consecutive k -free integers, which will be the central motivation of this paper. More precisely, for integers $k \geq 2$ and $h \geq 1$, let $N_{k,h}(x)$ be the number of integers $n \leq x$ such that both n and $n + h$ are k -free. Then, we have the following:

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Theorem 1. *For all $\epsilon > 0$ and all sufficiently large x , we have that*

$$N_{k,h}(x) = c_{k,h}x + O_{\epsilon,k,h}(x^{\omega(k)+\epsilon}),$$

where

$$c_{k,h} = \prod_p \left(1 - \frac{\rho_{k,h}(p)}{p^k}\right),$$

with

$$\rho_{k,h}(p) = \begin{cases} 2 & \text{if } p^k \nmid h \\ 1 & \text{if } p^k \mid h \end{cases},$$

and

$$\omega(k) = \begin{cases} (26 + \sqrt{433})/81 \approx 0.578 & \text{if } k = 2 \\ 25/64 \approx 0.391 & \text{if } k = 3 \end{cases},$$

and

$$\omega(k) = \frac{4k^2 + 13k + 9 - \sqrt{16k^4 - 40k^3 + 97k^2 + 234k + 81}}{9k(k+1)} \quad \text{for } k \geq 4.$$

The problem of estimating $N_{k,h}(x)$ as in Theorem 1 is most interesting when k is fairly small and it is this case where our methods work particularly well. We note that the error term $O(x^{2/(k+1)+\epsilon})$ can be obtained with elementary arguments (see for example Carlitz [3]). We also remark upon the elementary fact that

$$S_k(x) := \#\{n \leq x : n \text{ is } k\text{-free}\} = \frac{x}{\zeta(k)} + O(x^{1/k}).$$

By considering the Dirichlet series of $\zeta(s)/\zeta(ks)$, it is plausible that the error term of $S_k(x)$ cannot be improved by a proper power of x below $x^{1/k}$ without assuming some quasi-Riemann-Hypothesis. By assuming the error term of Theorem 1 is uniform in h , we may relate Theorem 1 to the error term of $S_k(x)$ and it is plausible to assume that we cannot get the error term of Theorem 1 below $x^{1/k}$ by a power of x without radically new ideas. Heath-Brown [6] considered the problem of Theorem 1 in the case $k = 2$ and $h = 1$. He obtained the error term $O(x^{7/11+\epsilon})$ ($7/11 \approx 0.636$) improving the trivial bound $O(x^{2/3+\epsilon})$. Heath-Brown's approach uses exponential sums and the Square Sieve. It should be noted that Heath-Brown's method is actually uniform in h . Brandes [2] has generalized Heath-Brown's method for general k . The error term she gets is $O(x^{14/(7k+8)+\epsilon})$ which is the currently best available result in the literature. Our error exponent $\omega(k)$ stated in Theorem 1 is non-trivial for all k and we have $\omega(k) \leq 14/(7k+8)$ for $k \leq 17$, with particularly good improvements in the cases $k = 2, 3$. For $k = 3$, the trivial exponent is $1/2$, and $14/(7k+8) \approx 0.483$, and we obtain $\omega(3) \approx 0.391$.

The strategy of our proof of Theorem 1 will be to reduce the problem of counting integer points on the algebraic variety $e^k v - d^k u = h$ inside a certain box. This will then be the same as counting rational points "close" to the curve $t = s^2$ where $t = v/u$ and $s = d/e$. The fundamental tool we use to tackle this counting problem will be the Determinant Method for which the interested reader should consult [8]. Indeed, our proof of Theorem

1 is very similar in many stages to Heath-Brown [10] where he derives an asymptotic formula for square-free values of the form $n^2 + 1$.

By using machinery developed in the proof for Theorem 1, we will prove the following theorem:

Theorem 2. *Let $k \in \{2, 3\}$, $r \geq 2$ and $l_i(x) = a_i x + b_i \in \mathbb{Z}[x]$ for $i = 1, \dots, r$ such that $a_i b_j - a_j b_i \neq 0$ and $a_i \neq 0$ for all i, j with $1 \leq i, j \leq r$ and $i \neq j$. Then define*

$$\rho(p) = \# \left\{ n \pmod{p^k} : p^k \mid l_i(n) \text{ for some } i \right\},$$

and let

$$c = \prod_p \left(1 - \frac{\rho(p)}{p^k} \right).$$

If $N(x)$ is the number of integers $n \leq x$ such that $l_1(n), \dots, l_r(n)$ are all k -free. Then for any $\epsilon > 0$ and any sufficiently large x we have that

$$N(x) = cx + O_\epsilon(x^{3/(2k+1)+\epsilon}).$$

It is not difficult to extend Theorem 2 to arbitrary k . For $k \geq 4$, the appropriate error term will be $O(x^{\omega(k)+\epsilon})$, where $\omega(k)$ is as stated in Theorem 1.

It should be noted that the implied constant in Theorem 2 depends on the choice of the l_i and that the best error term in the sense of Theorem 2 available in the literature for $k = 2$ was $O(x^{7/11+\epsilon})$ (See Tsang [11]). Tsang's proof uses a form of the Rosser-Iwaniec sieve and the version of Theorem 1 due to Heath-Brown. It should be noted that even though Tsang's error term is weaker than ours, his implied constants are uniform in r and $\max_i \|l_i\|$.

Finally, we will prove the following theorem about consecutive square-full numbers. Recall that an integer n is square-full if for all primes $p \mid n$, we have $p^2 \mid n$.

Theorem 3. *Let $N(x)$ be the number of integers $n \leq x$ such that both n and $n + 1$ are square-full. Then we have for all $\epsilon > 0$ and sufficiently large x that*

$$N(x) \ll_\epsilon x^{29/100+\epsilon}.$$

It can be shown that there are indeed infinitely many consecutive square-full numbers, for if n and $n + 1$ are square-full then so are $4n(n + 1)$ and $4n(n + 1) + 1$. However, it follows from a simple application of the abc-conjecture that there are at most finitely many n such that $n, n + 1$ and $n + 2$ are all square-full and so it does not seem interesting to have an equivalent of Theorem 2 for square-full numbers. The proof of Theorem 3 will be based on counting integer points on the variety $e^3 v^2 - d^3 u^2 = 1$ which will then lead to an argument very similar to the proof of Theorem 1. It is not difficult to extend the proof of Theorem 3 to consecutive k -full numbers but the error terms for $k > 2$ do not appear to be very good.

2 The Proof of Theorem 1

2.1 Preliminaries

First, we illustrate on how finding an asymptotic formula for $N_{k,h}(x)$ can be reduced to counting points on the algebraic variety $e^k v - d^k u = h$ inside a certain bounded box. In what follows all implied constants may depend on h and k . First observe that by dyadic subdivision it is enough to show that

$$N_{k,h}(2x) - N_{k,h}(x) = c_{k,h}x + O_{h,k,\epsilon}(x^{\omega(k)+\epsilon}).$$

Let

$$\xi(n) = \left(\prod_{p^k | n} p \right) \left(\prod_{p^k | n+h} p \right).$$

Then $\xi(n) = 1$ if and only if n and $n+h$ are k -free. Thus, we may deduce that

$$\begin{aligned} N_{k,h}(2x) - N_{k,h}(x) &= \sum_{x < n \leq 2x} \sum_{m | \xi(n)} \mu(m) \\ &= \sum_{m=1}^{\infty} \mu(m) N(x; m), \end{aligned}$$

where

$$N(x; m) = \# \{x < n \leq 2x : \xi(n) \equiv 0 \pmod{m}\}.$$

Observe that the congruence $\xi(n) \equiv 0 \pmod{p}$ has exactly $\rho_{k,h}(p)$ solutions modulo p^k . Thus, by an argument similar to the proof of the Chinese remainder theorem, the congruence $\xi(n) \equiv 0 \pmod{m}$ has exactly

$$\rho_{k,h}(m) = \prod_{p|m} \rho_{k,h}(p)$$

solutions modulo m^k . Hence

$$N(x; m) = \rho_{k,h}(m) \left(\frac{x}{m^k} + O(1) \right).$$

Note that for square-free m , we have that

$$\rho_{k,h}(m) \leq 2^{\omega(m)} \ll m^{\epsilon}.$$

Next, we introduce a parameter y with $x^{1/k} \leq y \leq x^{2/(k+1)}$. Later, we will pick y depending on k . Now, we look at the small terms in the above sum corresponding to

the values of m with $m \leq y$. These terms contribute

$$\begin{aligned}
& \sum_{m \leq y} \mu(m) \rho_{k,h}(m) \left(\frac{x}{m^k} + O(1) \right) \\
&= x \sum_{m \leq y} \frac{\mu(m) \rho_{k,h}(m)}{m^k} + O \left(\sum_{m \leq y} \rho_{k,h}(m) \right) \\
&= x \sum_{m=1}^{\infty} \frac{\mu(m) \rho_{k,h}(m)}{m^k} + O \left(x \sum_{m > y} \frac{\rho_{k,h}(m)}{m^k} + \sum_{m \leq y} \rho_{k,h}(m) \right) \\
&= c_{k,h} x + O(x^{1+\epsilon} y^{1-k}) + O(x^\epsilon y) \\
&= c_{k,h} x + O(x^\epsilon y),
\end{aligned}$$

where the last equality follows from $x^{1/k} \leq y$. Thus, we can see that the values of m with $m \leq y$ contribute our main term. Hence, we are left to consider the values of m with $m > y$. For each such m we write $m = de$ where $d^k \mid n$ and $e^k \mid n + h$. By dyadic subdivision, these values d, e lie in $O((\log x)^2)$ boxes $D/2 < d \leq D$, $E/2 < e \leq E$ where $D, E \ll x^{1/k}$ and $DE \gg y$. Hence, for one such pair D, E we must have

$$N_{k,h}(2x) - N_{k,h}(x) = c_{k,h} x + O(yx^\epsilon) + O(x^\epsilon \mathcal{N}(D, E)),$$

where

$$\mathcal{N}(D, E) = \#\{(d, e, u, v) \in \mathbb{N}^4 : D/2 < d \leq D, E/2 < e \leq E, x < d^k u + h = e^k v \leq 2x\}.$$

Thus, it remains to find an upper bound for $\mathcal{N}(D, E)$. We therefore have to study the solutions of the Diophantine equation

$$e^k v - d^k u = h. \tag{1}$$

We may also assume that $e^k v$ and $d^k u$ are coprime since the equation (1) can be reduced to $O_{k,h}(1)$ equations of the same type with the additional condition that $(d^k u, e^k v) = 1$. We will set $U := \frac{x}{D^k}$ and $V := \frac{x}{E^k}$ so that $u \asymp U$, $e \asymp E$, $d \asymp D$ and $v \asymp V$. Our overall goal is to pick y as small as possible so that we can still show $\mathcal{N}(D, E) \ll y$. This will then result in an overall error term $O(yx^\epsilon)$ in Theorem 1.

Next, we will derive a trivial estimate for $\mathcal{N}(D, E)$ by first summing over the pairs e, u

as follows:

$$\begin{aligned}
\mathcal{N}(D, E) &\ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \# \left\{ d : D/2 < d \leq D, e^k v - d^k u = h, (e, u) = 1 \right\} \\
&\ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \# \left\{ d : D/2 < d \leq D, d^k \equiv -hu^{-1} \pmod{e^k} \right\} \\
&\ll \sum_{\substack{E \ll e \ll E \\ U \ll u \ll U}} \left(\frac{D}{e^k} + 1 \right) \# \left\{ d \pmod{e^k}, d^k \equiv -hu^{-1} \pmod{e^k} \right\} \\
&\ll EU \left(\frac{D}{E^k} + 1 \right) x^\epsilon = \left(\frac{x}{(DE)^{k-1}} + EU \right) x^\epsilon \\
&\ll \max(y, EU) x^\epsilon,
\end{aligned} \tag{2}$$

where the last estimate follows again from $x^{1/k} \leq y \ll DE$. Thus, we may assume that $EU \geq y$. By interchanging D with E and U with V , we can analogously assume that $DV \geq y$. Next, observe that for fixed u, v :

$$\# \{ (d, e) : D/2 < d \leq D, E/2 < e \leq E, e^k v - d^k u = h \} \ll x^\epsilon. \tag{3}$$

This follows since $e^k v - d^k u = h$ is a Thue equation for $k \geq 3$ which has only a finite number of solutions (Thue [12]). For $k = 2$, the estimate (3) follows from Estermann [4]. Thus, we obtain the estimate $\mathcal{N}(D, E) \ll UV x^\epsilon$ and hence we may assume that $y \ll DE \leq x^{2/k} y^{-1/k}$.

Next, we will set $t = v/u$ and $s = d/e$ so that we can approximate the equation $e^k v - d^k u = h$ by

$$t = s^k + O\left(\frac{D^k}{E^k x}\right). \tag{4}$$

Thus, we have transformed our problem of counting integer points on a three-fold into a problem where we count rational points close to the curve $t = s^k$ where the sizes of the numerators and denominators of s and t are determined by D and E . The Determinant Method seems to be stronger in counting-problems involving varieties of lower dimension so that dealing with a curve rather than with a three-fold will provide the key saving in our proof. In section 2.2 we will show how the Determinant Method allows us to subdivide the range of s into intervals I of equal length so that our problem transforms into counting rational points close to the curve $t = s^k$ where s belongs to some particular interval I . In section 2.3 we will then calculate the contribution of one such interval I to $\mathcal{N}(D, E)$ and in section 2.4 we will then add up all the contributions to get the error exponent $\omega(k)$ for $k \geq 3$ and $7/12$ for $k = 2$. Finally, in section 2.5, we will further improve the error exponent for $k = 2$ by counting the points on lines contained in the three-fold $e^2 v - d^2 u = h$. This will allow us to get the error term $O(x^{\omega(2)+\epsilon})$ stated in Theorem 1. It will become clear in section 2.4 why, due to a technical restriction of the proof, the error term for $k \geq 4$ takes a different shape and is worse than the good error terms for $k = 2, 3$.

2.2 Determinant Method

Let $0 < m < 1$ be a real number which we will choose later. Note that s is of exact order D/E , so we may pick an integer $M \in [x^m, x]$ and split the range of s into $O(M)$ intervals $I = (s_0, s_0(1 + M^{-1})]$. For the rest of this section we will fix one such interval I and consider solutions (s, t) of (4) with $s \in I$. We label these solutions as $(s_1, t_1), \dots, (s_J, t_J)$ say. Consider one such (s_j, t_j) . We can write $s_j = s_0(1 + \alpha_j)$ where $0 < \alpha_j \leq 1/M$ and

$$t_j = s_j^k + O\left(\frac{D^k}{E^k x}\right) = s_0^k \left((1 + \alpha_j)^k + O\left(\frac{1}{x}\right) \right).$$

Hence, we can write

$$\begin{aligned} s_j &= s_0(1 + \alpha_j) && \text{with } 0 < \alpha_j \leq \frac{1}{M}, \\ t_j &= s_0^k(1 + p(\alpha_j) + \beta_j) && \text{with } \beta_j \ll \frac{1}{x}, \end{aligned}$$

where $p(\alpha_j)$ is a polynomial in α_j with no constant coefficient and with coefficients of size $O(1)$. The next step is to choose positive integers A and B and to label the monomials $s^a t^b$ with $a \leq A$ and $b \leq B$ as $m_1(s, t), \dots, m_H(s, t)$, where $H = (A + 1)(B + 1)$. Then one considers the $J \times H$ matrix \mathcal{M} whose (j, h) -th entry is $m_h(s_j, t_j)$. We will show that the rank of \mathcal{M} is strictly less than H provided we choose A, B and M appropriately. This will enable us to deduce that there is a non-zero vector \mathbf{c} such that $\mathcal{M}\mathbf{c} = 0$. The vector \mathbf{c} can be constructed from subdeterminants of \mathcal{M} which shows that $\mathbf{c} \in \mathbb{Q}^H$ has rational entries with numerators and denominators of size $\ll_{K,L} x^{H(A+B)}$ since s_j and t_j have numerators and denominators of size $\ll x$. If we now consider the polynomial $C_I(s, t) = C(s, t) = \sum_{h=1}^H c_h m_h(s, t)$ then we can see that $C(s_j, t_j) = 0$ for all our solutions with $s_j \in I$. By clearing out the common denominator of the coefficients of C we may assume that C has integer coefficients of size $\ll x^{H^2(A+B)}$.

To show that \mathcal{M} has rank strictly less than H we can assume that $H \leq J$ since otherwise this gets trivial. Thus, it suffices to show that every $H \times H$ subdeterminant of \mathcal{M} , vanishes. Without loss of generality it suffices to show that the determinant Δ coming from the first H rows and columns of \mathcal{M} vanishes. The j -th row of \mathcal{M} has entries with common denominator $e_j^A u_j^B$ which implies that

$$\left(\prod_{j \leq H} e_j^A u_j^B \right) \Delta \in \mathbb{Z}. \tag{5}$$

Observe that $\prod_{j \leq H} e_j^A u_j^B \leq E^{AH} U^{BH}$ and hence if we can show

$$|\Delta| < E^{-AH} U^{-BH}$$

then the integer in (5) has to be an integer strictly less than 1 which implies $\Delta = 0$. We substitute $s_j = s_0(1 + \alpha_j)$ and $t_j = s_0^k(1 + p(\alpha_j) + \beta_j)$ so that \mathcal{M} has entries

$$s_0^{a+kb}(1 + \alpha_j)^a(1 + p(\alpha_j) + \beta_j)^b.$$

Hence

$$\Delta = \left(\prod_{a=0}^A \prod_{b=0}^B s_0^{a+kb} \right) \Delta_1 = s_0^{\frac{1}{2}H(A+kB)} \Delta_1,$$

where Δ_1 is the determinant of the generalized Vandermonde matrix with its entries being polynomials in α_j and β_j and coefficients of size $O_{A,B}(1)$. Note that we have

$$|\alpha_j| \leq X_1^{-1} \text{ and } |\beta_j| \leq X_2^{-1},$$

where X_1 is of exact order M and X_2 is of exact order x . In particular, note that $\log X_1$ and $\log X_2$ are both of exact order $\log x$. We now order the monomials $X_1^{-a} X_2^{-b}$ decreasing in size, $1 = M_0, M_1, \dots, M_H, \dots$ say. Then by Lemma 3 in Heath-Brown [9], we may bound Δ_1 and hence Δ as follows:

$$\Delta = s_0^{\frac{1}{2}H(A+kB)} \Delta_1 \ll_{K,L} (D/E)^{\frac{1}{2}H(A+kB)} \prod_{h=1}^H M_h.$$

Let $M_H = W^{-1}$. Then $X_1^{-a} X_2^{-b} \geq M_H$ if and only if

$$a \log X_1 + b \log X_2 \leq \log W. \quad (6)$$

We want to count the number of pairs (a, b) which satisfy this inequality. We set $Q := \frac{\log X_2}{\log X_1}$ and $R := \frac{\log W}{\log X_1}$ so that the inequality can be written as $a + bQ \leq R$. Note that a can take the values $0, 1, \dots, \lfloor R \rfloor$ and for each a in this range, b can take the values $0, 1, \dots, \left\lfloor \frac{R-a}{Q} \right\rfloor$. So the number of pairs (a, b) satisfying this inequality is

$$\begin{aligned} \sum_{a=0}^{\lfloor R \rfloor} \left(\left\lfloor \frac{R-a}{Q} \right\rfloor + 1 \right) &= \sum_{a=0}^{\lfloor R \rfloor} \left(\frac{R-a}{Q} + O(1) \right) \\ &= \frac{R}{Q} (\lfloor R \rfloor + 1) - \frac{1}{Q} \sum_{a=0}^{\lfloor R \rfloor} a + O(\lfloor R \rfloor + 1) \\ &= \frac{1}{2} \frac{R^2}{Q} + O\left(\frac{R}{Q}\right) + O(R) + O(1) \\ &= \frac{1}{2} \frac{(\log W)^2}{(\log X_1)(\log X_2)} + O\left(\frac{\log W}{\log X_2}\right) + O\left(\frac{\log W}{\log X_1}\right) + O(1) \\ &= \frac{1}{2} \frac{(\log W)^2}{(\log X_1)(\log X_2)} + O\left(\frac{\log W}{\log x}\right) + O(1). \end{aligned}$$

The number of such pairs must be H which gives

$$|2H(\log X_2)(\log X_1) - (\log W)^2| \ll (\log W)(\log x) + (\log x)^2,$$

and hence

$$|\sqrt{2H(\log X_2)(\log X_1)} - \log W| \ll (\log x) \frac{(\log W) + (\log x)}{\sqrt{2H(\log X_2)(\log X_1)} + \log W} \ll \log x,$$

since $(\log X_2)(\log X_1) \asymp (\log x)^2$. Therefore

$$\log W = \sqrt{2H(\log X_2)(\log X_1)} + O(\log x). \quad (7)$$

Next, observe that

$$\log \prod_{h=1}^H M_h = - \sum_{a,b} (a \log X_1 + b \log X_2),$$

where the summation is subject to the inequality (6). First note that

$$\begin{aligned} \sum_{a,b} a &= \sum_{a=0}^{\lfloor R \rfloor} \sum_{b=0}^{\lfloor \frac{R-a}{Q} \rfloor} a \\ &= \sum_{a=0}^{\lfloor R \rfloor} a \left(\frac{R-a}{Q} + O(1) \right) \\ &= \frac{R}{Q} \left(\frac{1}{2} R^2 + O(R) + O(1) \right) - \frac{1}{Q} \left(\frac{1}{3} R^3 + O(R^2) + O(1) \right) + O(R^2) \\ &= \frac{1}{6} \frac{R^3}{Q} + O \left(\frac{R^2}{Q} \right) + O(R^2) + O(1) \\ &= \frac{1}{6} \frac{(\log W)^3}{(\log X_1)^2 (\log X_2)} + O \left(\left(\frac{\log W}{\log x} \right)^2 \right) + O(1). \end{aligned}$$

Similarly, one may evaluate

$$\sum_{a,b} b = \frac{1}{6} \frac{(\log W)^3}{(\log X_2)^2 (\log X_1)} + O \left(\left(\frac{\log W}{\log x} \right)^2 \right) + O(1),$$

which gives

$$\log \prod_{h=1}^H M_h = - \frac{1}{3} \frac{(\log W)^3}{(\log X_1)(\log X_2)} + O \left(\frac{(\log W)^2}{\log x} \right) + O(\log x).$$

Substituting in (7) yields

$$\log \prod_{h=1}^H M_h = - \frac{2\sqrt{2}}{3} H^{\frac{3}{2}} \sqrt{(\log X_1)(\log X_2)} + O(H \log x).$$

This shows that

$$\log |\Delta| \leq O_{A,B}(1) + \frac{1}{2} H(A + kB) \log(D/E) - \frac{2\sqrt{2}}{3} H^{\frac{3}{2}} \sqrt{(\log X_1)(\log X_2)} + O(H \log x).$$

Thus, to get our required bound on $\log |\Delta|$ it suffices to show that

$$A \log E + B \log U + \frac{1}{2} (A + kB) \log \frac{D}{E} + C_1(A, B) + C_2 \log x \leq \frac{2\sqrt{2}}{3} H^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)},$$

where $C_1(A, B)$ is some constant depending on A and B and C_2 is an absolute constant. Note that $AB \leq H$ and

$$A \log E + B \log U + \frac{1}{2}(A + kB) \log \frac{D}{E} = \frac{A}{2} \log(DE) + \frac{B}{2} \log(UV).$$

Hence it suffices to show

$$\frac{1}{2}A \log(DE) + \frac{1}{2}B \log(UV) + C_1(A, B) + C_2 \log x \leq \frac{2\sqrt{2}}{3}(AB)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)}.$$

We will optimize the error by choosing $A = \left\lfloor \frac{B \log(UV)}{\log(DE)} \right\rfloor$. Recall that DE is bounded below and above by powers of x which implies that $\nu := \frac{\log(UV)}{\log(DE)} \asymp 1$. In particular, if $B \geq 2/\nu$ then we may apply the inequalities $\frac{1}{2}B\nu \leq \lfloor B\nu \rfloor \leq B\nu$ for $B \geq 2/\nu$ to get that $A \asymp B$. Analyzing Lemma 3 of Heath-Brown [9] which produces our constant $C_1(A, B)$, we can see that $C_1(A, B)$ is actually a polynomial in A and B . This enables us to replace $C_1(A, B)$ by some $C_3(B)$ say. Now observe that $\frac{1}{2}A \log(DE) + \frac{1}{2}B \log(UV) \leq B \log(UV)$ and that

$$\begin{aligned} \frac{2\sqrt{2}}{3}(AB)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)} &\geq \frac{2\sqrt{2}}{3}B\nu^{\frac{1}{2}} \left(1 - \frac{1}{B\nu}\right)^{\frac{1}{2}} \sqrt{(\log X_1)(\log X_2)} \\ &\geq \frac{2\sqrt{2}}{3}B\nu^{\frac{1}{2}} \left(1 - \frac{1}{B\nu}\right) \sqrt{(\log X_1)(\log X_2)} \\ &= \frac{2\sqrt{2}}{3}B \sqrt{\frac{\log(UV)}{\log(DE)}} \sqrt{(\log X_1)(\log X_2)} + O(\log x) \end{aligned}$$

since $(\log X_1)(\log X_2) \asymp (\log x)^2$. It therefore suffices if we have

$$B \log(UV) + C_3(B) + C_2 \log x \leq \frac{2\sqrt{2}}{3}B \sqrt{\frac{\log(UV)}{\log(DE)}} \sqrt{(\log X_1)(\log X_2)}.$$

Let $\delta > 0$ be arbitrary and pick X_1 such that

$$\frac{2\sqrt{2}}{3} \sqrt{\frac{\log(UV)}{\log(DE)}} \sqrt{(\log X_1)(\log X_2)} \geq (1 + \delta) \log(UV),$$

whence it suffices if we have $B \log(UV) + C_3(B) + C_2 \log x \leq B(1 + \delta) \log(UV)$ which holds if and only if $B\delta \log(UV) \geq C_3(B) + C_2 \log x$. Pick

$$B = B(\delta) \geq \frac{2C_2 \log x}{\delta \log(UV)}$$

so that $C_2 \log x \leq \frac{1}{2}\delta B \log(UV)$ and pick x large enough in terms of δ so that $C_3(B) \leq \frac{1}{2}\delta B \log(UV)$. Thus, we have shown that if we pick B (and hence A) and x large enough

in terms of δ and if we can pick a suitable X_1 as above then $\Delta = 0$. Observing that $X_1 \gg M$ and $X_2 \gg x$ We may rewrite the condition with a redefined δ as

$$\log M \geq \frac{9}{8}(1 + \delta) \frac{\log(UV) \log(DE)}{\log x}.$$

In summary, we get the following lemma:

Lemma 4. *Let $\delta > 0$, $0 < m < 1$ and assume that $M \in [x^m, x]$ satisfies*

$$\log M \geq \frac{9}{8}(1 + \delta) \frac{\log(DE) \log(UV)}{\log x}.$$

Then for any interval $I = [s_0, s_0(1 + M^{-1})]$, there exists a non-zero integer polynomial $C_I(s, t)$ of total degree $O_\delta(1)$ and coefficients of size $O(x^\kappa)$ where $\kappa = \kappa(\delta)$ and such that $C_I(s, t) = 0$ for all $s \in I$.

2.3 Counting Solutions

We will start by showing that C_I can be assumed to be absolutely irreducible. Let F be a monic factor of C_I which is not defined over \mathbb{Q} and consider a rational point (s, t) such that $F(s, t) = 0$. Then for every conjugate F^σ of F we have that $F^\sigma(s, t) = 0$. Thus, the number of possible points (s, t) is $O_\delta(1)$ by Bézout's Theorem. Since d, e and u, v are coprime we get $O_\delta(1)$ solutions for such factors. Thus, it suffices to consider absolutely irreducible factors F of C_I defined over \mathbb{Z} . The height of such an F is still bounded by a power of x by Gelfond's Lemma (Bombieri and Gubler [1, Lemma 1.6.11]). The number of different factors is $O_\delta(1)$. Thus it suffices to consider one absolutely irreducible factor of C_I which satisfies the same conditions as C_I . By clearing the denominators of $F(s, t) = 0$ we may rewrite the equation in the form

$$F(d, e; v, u) = 0. \tag{8}$$

We now want to estimate how much a fixed interval I contributes to $\mathcal{N}(D, E)$. We have so far established that a solution (d, e, u, v) of (1) with $s \in I$ must satisfy (8). The condition $s \in I$ gives $|d - es_0| \leq D/M$ since $|e| \leq E$. It is convenient to define the linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(x_1, x_2) = \left(\frac{M}{D}(x_1 - x_2 s_0), \frac{1}{E}x_2 \right).$$

This linear map defines a lattice

$$\Lambda = \{T(x_1, x_2) : (x_1, x_2) \in \mathbb{Z}^2\}$$

of determinant $\det(\Lambda) = M/(ED)$. Consider the square

$$S = \{(\alpha_1, \alpha_2) : |\alpha_1|, |\alpha_2| \leq 1\}.$$

We know that for each $s = d/e \in I$, we must have $T(d, e) \in \Lambda \cap S$. Thus, we will now count points falling into $\Lambda \cap S$.

Let $\mathbf{g}^{(1)}$ be the shortest non-zero vector in Λ and let $\mathbf{g}^{(2)}$ be the shortest vector in Λ not parallel to $\mathbf{g}^{(1)}$. Then $\mathbf{g}^{(1)}, \mathbf{g}^{(2)}$ will be a basis for Λ . Moreover, $\lambda_1 \mathbf{g}^{(1)} + \lambda_2 \mathbf{g}^{(2)} \in S$ implies $|\lambda_1 \mathbf{g}^{(1)}| \ll 1$ and $|\lambda_2 \mathbf{g}^{(2)}| \ll 1$. By defining L_i to be a suitable constant times $|\mathbf{g}^{(i)}|^{-1}$ for $i = 1, 2$ we may write $|\lambda_i| \leq L_i$. Note that $|\mathbf{g}^{(1)}| \leq |\mathbf{g}^{(2)}|$ and $|\mathbf{g}^{(1)}| |\mathbf{g}^{(2)}| \ll \det(\Lambda) = M/(DE)$ from which we may conclude $L_2 \ll L_1$ and $L_1 L_2 \gg \frac{DE}{M}$. Next set

$$\mathbf{h}^{(i)} = \left(\frac{D}{M} \mathbf{g}_1^{(i)} + s_0 E \mathbf{g}_2^{(i)}, E \mathbf{g}_2^{(i)} \right).$$

Then $\mathbf{h}^{(1)}$ and $\mathbf{h}^{(2)}$ will form a basis for \mathbb{Z}^2 and if $\mathbf{x} = \lambda_1 \mathbf{h}^{(1)} + \lambda_2 \mathbf{h}^{(2)}$ is in the region given by $|x_1 - x_2 s_0| \leq D/M$ and $|x_2| \leq E$ then $\lambda_1 \mathbf{g}^{(1)} + \lambda_2 \mathbf{g}^{(2)} \in S$. Thus, after a change of basis we may replace (d, e) by (λ_1, λ_2) with $|\lambda_i| \leq L_i$ where $L_2 \ll L_1$ and $L_1 L_2 \gg (DE)/M$.

Our value s_0 is of the form $s_0 = \frac{x_3 D}{M E}$ where x_3 is an integer of exact order M . Define $x_4 := \lfloor x_3^k M^{1-k} \rfloor$ and $t_0 := \frac{x_4 V}{M U}$ so that $s_0^k = t_0 + O\left(\frac{1}{M} \frac{V}{U}\right)$. Observe that

$$t = s_0^k + s_0^k p(\alpha) + s_0^k \beta = s_0^k + O\left(\frac{1}{M} \frac{V}{U}\right) + O\left(\frac{1}{x} \frac{V}{U}\right) = t_0 + O\left(\frac{1}{M} \frac{V}{U}\right),$$

since $M \leq x$. This leads to the conditions $|v - ut_0| \ll \frac{V}{M}$ and $|u| \leq U$ and hence we can analogously to the above argument replace (v, u) by (τ_1, τ_2) say where $|\tau_i| \leq T_i$ with $T_2 \ll T_1$ and $T_1 T_2 \gg \frac{UV}{M}$. These substitutions convert equation (1) into an equation

$$G_0(\lambda_1, \lambda_2; \tau_1, \tau_2) = h, \quad (9)$$

say where G_0 is bi-homogeneous. Similarly, equation (8) will turn into an equation of the form

$$G_1(\lambda_1, \lambda_2; \tau_1, \tau_2) = 0 \quad (10)$$

where G_1 is bi-homogeneous of degree (a, b) say and satisfies the same conditions as F . Also, from the above argument it is clear that the vectors (λ_1, λ_2) and (τ_1, τ_2) are primitive. For example, if $p \mid (\lambda_1, \lambda_2)$ then $p \mid (d_1, e_1) = 1$.

If $a = 0$ then (10) determines $O_\delta(1)$ pairs (τ_1, τ_2) each of which gives a pair (u, v) . By (3), the number of pairs (d, e) corresponding to such a (u, v) is $\ll x^\delta$. Thus, the case $a = 0$ contributes $O_\delta(x^\delta)$ to $\mathcal{N}(D, E)$.

Similarly, if $b = 0$ then (10) determines $O_\delta(1)$ pairs (λ_1, λ_2) each of which gives a pair (d, e) . Note that for each such pair (d, e) , the equation (1) is a linear Diophantine equation in (u, v) and thus, for each pair (d, e) , the number of solutions (u, v) to (1) is

$$\ll \frac{V}{D^k} + 1 = \frac{x}{(DE)^k} + 1 \ll 1,$$

where the last estimate follows again from $x^{1/k} \leq y \ll DE$. Hence, the case $b = 0$ contributes $O_\delta(1)$ to $\mathcal{N}(D, E)$.

If $a \geq 2$ then Lemma 2 of Heath-Brown [10] gives us that (10) has $O_{\epsilon, \delta}(T_1^{1+\epsilon} x^{\kappa \epsilon})$ solutions. (recall that the coefficients of G_1 are of order x^κ). Picking ϵ small enough in terms of δ , we may assume that this is $O_\delta(T_1^{1+\delta} x^\delta)$. Each solution of (10) produces at most one solution of (1). So in the case $a \geq 2$ the contribution of I to $\mathcal{N}(D, E)$ is $O_\delta(T_1^{1+\delta} x^\delta)$. Similarly, if $b \geq 2$ then the contribution of I to $\mathcal{N}(D, E)$ is $O_\delta(L_1^{1+\delta} x^\delta)$.

If $a = 1$ then (10) can be written as

$$\lambda_1 G_{11}(\tau_1, \tau_2) + \lambda_2 G_{12}(\tau_1, \tau_2) = 0$$

and hence $q\lambda_1 = G_{12}(\tau_1, \tau_2)$ and $q\lambda_2 = -G_{11}(\tau_1, \tau_2)$ for some integer q . We define two polynomials $g_1, g_2 \in \mathbb{Z}[x]$ by $G_{1i}(\tau_1, \tau_2) = \tau_1^b g_i(\tau_2/\tau_1)$ for $i = 1, 2$. Observe that g_1 and g_2 must be coprime since G_1 is absolutely irreducible. Hence, by the Euclid's Algorithm there exist polynomials $h_1, h_2 \in \mathbb{Z}[x]$ and an integer H such that

$$g_1 h_1 + g_2 h_2 = H$$

where $H = O(x^\kappa)$. Evaluating this equation at τ_2/τ_1 we may deduce that $q \mid H\tau_1^K$ where K is some integer. and similarly we may conclude that $q \mid \tilde{H}\tau_2^K$ where $\tilde{H} = O(x^\kappa)$. But τ_1 and τ_2 are coprime. Thus, we may deduce that we have $O(x^\delta)$ choices for q . Each value of q gives us a value of λ_1 and λ_2 in terms of τ_1 and τ_2 which we may substitute into (9) to get a Thue equation of the form $G_3(\tau_1, \tau_2) = hq^k$ say. This equation gives $O(T_1)$ possible pairs τ_1, τ_2 which shows that the case $a = 1$ contributes at most $O_\delta(x^\delta T_1)$ to $\mathcal{N}(D, E)$. Similarly, we may deduce that the case $b = 1$ contributes at most $O_\delta(x^\delta L_1)$ to $\mathcal{N}(D, E)$. In summary, we obtain the following:

Lemma 5. *For any $\delta > 0$, the contribution of the solutions (s, t) with $s \in I$ to $\mathcal{N}(D, E)$ is $O_\delta(x^\delta \min(L_1^{1+\delta}, T_1^{1+\delta}))$.*

2.4 Completion of the proof for $k \geq 3$

In the previous section, we calculated the contribution of each interval I to $\mathcal{N}(D, E)$. It remains to sum up the contribution of the various intervals. Here, our proof will break down in different cases according to the value of k . For convenience, let us write $y = x^r$ and $DE = x^\psi$ so that $UV = x^{2-k\psi}$. Recall that $k^{-1} \leq r \leq 2(k+1)^{-1}$. According to Lemma 4 we pick a value M such that

$$\frac{\log M}{\log x} = (1 + \delta) f_M(\psi),$$

with

$$f_M(\psi) = \frac{9}{8} \psi(2 - k\psi).$$

Recall that $y \ll DE \leq x^{2/k} y^{-1/k}$ so that essentially $r \leq \psi \leq k^{-1}(2 - r)$. Note that $f_M(\psi)$ is decreasing on the range of ψ under consideration. Thus, we may pick $m = f_M(k^{-1}(2 - r))/2$. Then $0 < m < 1$ and our choice of M will indeed satisfy $M \in$

$[x^m, x]$. In what follows, we will need that $\max(U/V, V/U) < M$. We will thus, derive a restriction on r which will ensure that this condition is satisfied. We will examine the inequality $V/U < M$ and we will fix ψ for now which will determine $f_M(\psi)$, the exponent of M . Observe that $V/U = x^{k\psi} E^{-2k}$, so that V/U gets largest when E is as small as possible. Note that

$$x^{k\psi} y \leq (DE)^k EU = E^{k+1} x,$$

since $EU \geq y$. Thus, $E \geq x^{\frac{k\psi+r-1}{k+1}}$ and the inequality $V/U < M$ is satisfied as long as

$$k\psi - \frac{2k}{k+1}(k\psi + r - 1) - f_M(\psi) < 0.$$

Note that the function on the left-hand side of this inequality has its maximum at $\psi = r$ when seen as a function of ψ . That is, $V/U < M$ holds as long as

$$kr - \frac{2k}{k+1}(kr + r - 1) - \frac{9}{8}r(2 - kr) < 0.$$

Note that this inequality is satisfied for $k = 2$ regardless of what value we pick for r . However, for $k \geq 3$, the inequality is only satisfied if $r \geq r_0(k) + \epsilon_1$ where

$$r_0(k) = \frac{4k^2 + 13k + 9 - \sqrt{16k^4 - 40k^3 + 97k^2 + 234k + 81}}{9k(k+1)}.$$

Thus, if we assume that $r \geq r_0(k) + \epsilon_1$, then $x^{\epsilon_2} V/U < M$. Observe that our argument was symmetric in D, E . Hence, if we impose the condition $r \geq r_0(k) + \epsilon_1$ on r then it follows that $x^{\epsilon_2} \max(D/E, E/D, U/V, V/U) < M$. We will use this fact in the argument below.

To proceed with summing up the contributions of the intervals, we write $\mathbf{g}^{(1)}$ from the previous section as

$$\mathbf{g}^{(1)} = ((M/D)(x_1 - x_2 s_0), (1/E)x_2)$$

and recall that $|L_1 \mathbf{g}^{(1)}| \ll 1$. This gives $L_1(x_1 - x_2 s_0) \ll D/M$ and $L_1 x_2 \ll E$. If $L_1 \gg E$ then $x_2 = 0$ and $x_1 \ll D/(ML_1) \ll E/L_1$ since $E/D < M$, which would also imply $x_1 = 0$. This is impossible since $(x_1, x_2) = 1$. Thus, we must have $L_1 \ll E$. Recall that we produce the intervals $I = (s_0, s_0 + \frac{1}{M} \frac{D}{E}]$ by taking $s_0 = x_3 \frac{1}{M} \frac{D}{E}$ for integers $x_3 \asymp M$. Hence, the number of intervals I for which $L < L_1 \leq 2L$ is at most the number of triples $(x_1, x_2, x_3) \in \mathbb{Z}^3$ for which $\gcd(x_1, x_2) = 1$ and

$$x_2 x_3 = \frac{ME}{D} x_1 + O\left(\frac{E}{L}\right), \quad x_2 \ll \frac{E}{L}, \quad x_3 \asymp M.$$

Recalling that $L_1 \gg L_2$ and $L_1 L_2 \gg (DE)/M$, we can deduce that $L \gg (DE/M)^{1/2}$. If $x_2 = 0$ then $x_1 = \pm 1$ since $(x_1, x_2) = 1$. Thus, we may deduce from the above equation that $ME/D \ll E/L$ which implies $M \ll D/E$. This is not possible for large

enough x since $x^\epsilon D/E < M$ by the above argument. Thus, the case $x_2 = 0$ cannot arise. Furthermore, we must have $x_3 \neq 0$ since x_3 has exact order M . Thus, we may assume that $x_2 x_3 \neq 0$. Then the above conditions on (x_1, x_2, x_3) imply that $x_1 \ll D/L$. Thus, there are $O(D/L + 1)$ choices for x_1 and each x_1 produces $O(E/L)$ choices for the product $x_2 x_3$. And each product $x_2 x_3$ gives $O_\delta(x^\delta)$ pairs (x_2, x_3) . Thus, there are $O_\delta(x^\delta(D/L + 1)(E/L))$ intervals I so that L_1 is of exact order L . Each interval contributes $O_\delta(x^\delta L^{1+\delta})$ by Lemma 5. Note that $L^\delta \ll E^\delta \ll x^\delta$ and hence we get a total contribution of these intervals of $O_\delta(x^{3\delta}(DE/L + E))$. Next we use that $L \gg (DE/M)^{1/2}$ and hence by dyadic subdivision, we can conclude that $\mathcal{N}(D, E) \ll_\delta x^{3\delta}(DEM)^{\frac{1}{2}}$ where we have used that $E \ll (DEM)^{1/2}$ since $E/D < M$.

Similarly, we may consider the (x_1, x_2) lattice from the previous section corresponding to (v, u) to get $T_1(x_1 - x_2 t_0) \ll V/M$ and $T_1 x_2 \ll U$ as well as $T_1 \ll U$. Recall that $t_0 = x_4 \frac{1}{M} \frac{V}{U}$ where $x_4 = \left\lfloor \frac{x_3^2}{M} \right\rfloor$. We can see that $x_3 \asymp M$ implies $x_4 \asymp M$ for large enough x . Hence by a completely analogous argument to the above we can deduce that $\mathcal{N}(D, E) \ll_\delta x^{3\delta}(UVM)^{\frac{1}{2}}$. Thus, we have established a bound on $\mathcal{N}(D, E)$:

$$\mathcal{N}(D, E) \ll_\delta x^{3\delta} M^{\frac{1}{2}} \min(DE, UV)^{\frac{1}{2}}.$$

Therefore,

$$\frac{\log \mathcal{N}(D, E)}{\log x} \leq 3\delta + \frac{1}{2} \min(\psi, 2 - k\psi) + \frac{9}{16} (1 + \delta) \psi (2 - k\psi).$$

Next, we observe that

$$\frac{1}{2} \min(\psi, 2 - k\psi) + \frac{9}{16} \psi (2 - k\psi) \leq \frac{1}{k+1} + \frac{9}{4} \frac{1}{(k+1)^2} = r_1(k),$$

say, with equality if $\psi = 2/(k+1)$. This completes the proof of Theorem 1 for $k \geq 3$: By setting $r = \max(r_0(k), r_1(k)) + \epsilon_1$, we obtain the desired error exponent $\omega(k) = r$. Note that for $k \geq 4$, we have $r_0(k) \geq r_1(k)$ while if $k = 3$, then $r_0(3) \leq r_1(3) = 25/64$ gives us a very good improvement over the trivial bound. For $k = 2$, the error exponent will be $r_1(2) = 7/12$. But we can do better than this by considering points on lines contained in the three-fold $e^2 v - d^2 u = h$. The following section will illustrate this idea.

2.5 Counting Points on Lines, Finishing the proof for $k = 2$

In this section, we will conclude the proof of Theorem 1 by considering points on lines contained in the three-fold (1). For convenience, we will illustrate the proof when $h = 1$ but with minor changes, the proof can be adapted to general h . We will set $m = 1/2$ and $A = B = 1$ in section 2.2. Note that in this case $M \in [x^{1/2}, x]$ and this will produce a 4×4 matrix \mathcal{M} where the j -th row is

$$\left(1 \quad s_0(1 + \alpha_j) \quad s_0^2(1 + 2\alpha_j + \beta_j) \quad s_0^3(1 + \alpha_j)(1 + 2\alpha_j + \beta_j) \right),$$

with $\alpha_j \ll M^{-1}$ and $\beta_j \ll x^{-1}$. (Note that $\alpha_j^2 \ll M^{-2} \leq x^{-1}$). Performing column operations, we get a matrix with j -th row

$$s_0^6 \cdot \begin{pmatrix} 1 & \alpha_j & \beta_j & \alpha_j(2\alpha_j + \beta_j) \end{pmatrix}.$$

Recalling that $s_0 \asymp D/E$, we may deduce that $\Delta \ll \frac{1}{M^3 x} \frac{D^6}{E^6}$. As before, we require $|\Delta| \ll E^{-4}U^{-4}$ in order to deduce that $\Delta = 0$. This is satisfied if $M > (xUV)^{1/3+\delta}$ where $\delta > 0$ is arbitrarily small and x is large enough in terms of δ . Setting as before $DE = x^\psi$ and $UV = x^{2-2\psi}$ we can see that $x^{1/2} \leq (xUV)^{1/3+\delta} \leq x$. This enables us to pick $M = (xUV)^{1/3+\delta}$ provided δ is small enough and x is large enough in terms of δ .

Hence, as before, the determinant method will produce an irreducible auxiliary polynomial $F(d, e; u, v)$. This polynomial will be bilinear since we picked the monomials in \mathcal{M} accordingly. That is, in section 2.3 the case that will occur is $a = b = 1$. Hence, (10) can be written as

$$dL_1(u, v) + eL_2(u, v) = 0,$$

where $L_1(u, v) = c_1u + c_2v$ and $L_2(u, v) = c_3u + c_4v$ are linear forms with integral coefficients. As in section 2.3, we get $O(x^\delta)$ choices for an integer q such that

$$eq = -L_1(u, v), \quad dq = L_2(u, v). \quad (11)$$

Plugging this into the equation $e^2v - d^2u = 1$ gives us a Thue equation

$$G_3(u, v) = (L_1(u, v))^2v - (L_2(u, v))^2u = q^2,$$

say. When G_3 is irreducible then this equation will only have a finite number of solutions (see Thue [12]). Thus, the equation will produce $O(x^\delta)$ solutions (u, v) for each q except if G_3 is splitting into three equal linear factors,

$$G_3(u, v) = \alpha(\alpha_1v - \alpha_2u)^3,$$

say where $\alpha, \alpha_1, \alpha_2 \in \mathbb{Z}$. Our aim is to show that the points under consideration corresponding to this case actually lie on a line contained in the three-fold defined by (1). Comparing coefficients we get the equations

$$\begin{aligned} \alpha\alpha_2^3 &= c_3^2 \\ \alpha\alpha_1^3 &= c_2^2 \\ 3\alpha\alpha_1^2\alpha_2 &= c_4^2 - 2c_1c_2 \\ 3\alpha\alpha_1\alpha_2^2 &= c_1^2 - 2c_3c_4. \end{aligned}$$

The first two equations give $\alpha_1 = (c_2^2/\alpha)^{1/3}$ and $\alpha_2 = (c_3^2/\alpha)^{1/3}$ which turns the third and fourth equation into

$$3c_2^{4/3}c_3^{2/3} = c_4^2 - 2c_1c_2, \quad \text{and} \quad (12)$$

$$3c_2^{2/3}c_3^{4/3} = c_1^2 - 2c_3c_4 \quad (13)$$

respectively. If $c_2 = 0$ then we may deduce from these equations that $c_4 = 0$ and hence $c_1 = 0$. Hence $L_1 = 0$ which gives a contradiction since $qe \neq 0$. Thus, $c_2 \neq 0$ and similarly $c_3 \neq 0$. Using (12) and (13) we may deduce that $c_3^{2/3}(c_4^2 - 2c_1c_2) = c_2^{2/3}(c_1^2 - 2c_3c_4)$ which may be written as

$$(c_3^{1/3}c_4 + c_3^{2/3}c_2^{2/3})^2 = (c_2^{1/3}c_1 + c_2^{2/3}c_3^{2/3})^2.$$

After taking the square-root, this either lets us deduce directly that $c_4^3c_3 = c_1^3c_2$ or that

$$c_3^{1/3}c_4 + c_2^{1/3}c_1 + 2c_2^{2/3}c_3^{2/3} = 0.$$

Multiply this equation by $c_3^{2/3}$ and plug in the expression for c_3c_4 given by (13) to get $(c_1 + c_2^{1/3}c_3^{2/3})^2 = 0$. Similarly, we may deduce from (12) that $(c_4 + c_3^{1/3}c_2^{2/3})^2 = 0$. This implies that in either case we have $c_4^3c_3 = c_1^3c_2$. Plugging this value of c_3 into (12) we get that

$$c_1 = \frac{1}{\kappa} \frac{c_4^2}{c_2}, \quad c_3 = \frac{1}{\kappa^3} \frac{c_4^3}{c_2^2},$$

where $\kappa \in \{-1, 3\}$. Consider the equation $c_4^3c_3 = c_1^3c_2$. We set $c_1 = h\alpha$ and $c_4 = h\beta$ where $h = (c_1, c_4)$. From this, we can deduce that $c_2 = k\beta^3$ and $c_3 = k\alpha^3$ for some integer k . Observe that $(h, k) = 1$ since $(c_1, c_2, c_3, c_4) = 1$ as F is irreducible. Considering the equation $c_4^2 = \kappa c_1 c_2$ we see that $h = \kappa k \alpha \beta$ which implies $k = 1$ and

$$c_1 = \kappa \alpha^2 \beta, \quad c_2 = \beta^3, \quad c_3 = \alpha^3, \quad c_4 = \kappa \alpha \beta^2.$$

Our original Thue equation has now turned into

$$v = \frac{\alpha^2}{\beta^2} u + \frac{Q^2}{\beta^2},$$

where $q = Q^3$ for $Q \in \mathbb{Z}$ with $Q > 0$. Substituting this expression for v into (11), we obtain that (d, e, u, v) must lie on the line

$$\ell : (d, e, u, v) = \left(\frac{\kappa \alpha}{Q}, -\frac{\beta}{Q}, 0, \frac{Q^2}{\beta^2} \right) + u \left((1 + \kappa) \frac{\alpha^3}{Q^3}, -(\kappa + 1) \frac{\alpha^2 \beta}{Q^3}, 1, \frac{\alpha^2}{\beta^2} \right).$$

Assume that the line does indeed have an integral point counted by $\mathcal{N}(D, E)$. Pick u_1 to be the smallest integer such that (d_1, e_1, u_1, v_1) is on ℓ and counted by $\mathcal{N}(D, E)$. This gives us the equations

$$Q^2 = v_1 \beta^2 - u_1 \alpha^2 \tag{14}$$

$$\alpha \kappa Q^2 + u_1 (1 + \kappa) \alpha^3 = d_1 Q^3 \tag{15}$$

$$-\beta Q^2 - u_1 (1 + \kappa) \alpha^2 \beta = e_1 Q^3. \tag{16}$$

We now consider the lines with $\kappa = -1$. In this case, we may deduce from (15) and (16) that $Q = 1$, $d_1 = -\alpha$ and $e_1 = -\beta$. Thus our points under consideration must lie on the line

$$(d, e, u, v) = (d_1, e_1, u_1, v_1) + \lambda(0, 0, \beta^2, \alpha^2),$$

where $\lambda \in \mathbb{Z}$. Note that there are no parallel lines with $\kappa = -1$ having the same α and β since (14) determines the pair (u_1, v_1) modulo (β^2, α^2) . The number of distinct lines with $\kappa = -1$ is therefore determined by the number of choices for the direction vectors, that is, by the number of choices for α and β . We must have $\alpha\beta \ll (UV)^{1/2} \ll x^{1/2}$ which shows that the number of distinct lines with $\kappa = -1$ is $O(x^{1/2+\delta})$. The number of points on each line is $\ll 1 + U/\beta^2 \ll 1$ since $E \asymp e_1 = -\beta$. Thus, the total number of points counted be $\mathcal{N}(D, E)$ lying on lines with $\kappa = -1$ is $O(x^{1/2+\delta})$.

Next, we consider the lines with $\kappa = 3$. From (15) and (16) we may deduce that $(\beta d_1 + \alpha e_1)Q = 2\alpha\beta$. Let $a = (Q, \alpha)$ and $b = (Q, \beta)$ so that $\alpha = aA, \beta = bB$ and $Q = abQ_1$ and $(Q_1, AB) = 1$ say. After canceling ab from the last equation, we may deduce that $A \mid d_1$ and $B \mid e_1$. Set $d_1 = AD_1$ and $e_1 = BE_1$. Finally, (14) implies that $a^2 \mid v_1$ and $b^2 \mid u_1$. Let $v_1 = a^2V_1$ and $u_1 = b^2U_1$. Furthermore, we can deduce from (14) that $(U_1, Q_1) = 1$ and hence (16) shows that $Q_1^2 \mid 4U_1A^2$ and therefore $Q_1^2 \mid 4$. The direction vector of the line ℓ is now

$$\left(\frac{A^3}{b^3}C, -\frac{A^2B}{ab^2}C, 1, \frac{a^2A^2}{b^2B^2} \right)$$

where $C = 4/Q_1^3 \in \{4, 1/2\}$. Note that a, A, b, B are pairwise coprime since $d_1^2u_1$ and $e_1^2v_1$ as well as α and β are coprime. Thus, all lines lying on the surface (1) can be written as

$$(AD_1, BE_1, b^2U_1, a^2V_1) + \lambda(aA^3B^2C, -bA^2B^3C, ab^3B^2, a^3bA^2)$$

where λ goes through $\mathbb{Z}, \mathbb{Z}/2$ or $2\mathbb{Z}$ and subject to the conditions

$$Q_1^2 = V_1B^2 - U_1A^2 \tag{17}$$

$$bD_1 + aE_1 = 2Q_1^{-1} \tag{18}$$

$$U_1A^2Q_1C - D_1bQ_1 = -3. \tag{19}$$

Fix Q_1 . Given a line ℓ , the direction vector and hence A, B, a, b are uniquely determined up to sign. Thus we also fix A and B and consider the number of points on lines with our fixed values of Q_1, A and B .

If D'_1, E'_1, U'_1, V'_1 is one solution of (17), (18), (19) then the other solutions D_1, E_1, U_1, V_1 are given by

$$D_1 = D'_1 + \mu aA^2B^2$$

$$E_1 = E'_1 - \mu bA^2B^2$$

$$U_1 = U'_1 + \mu abB^2/C$$

$$V_1 = V'_1 + \mu abA^2/C.$$

Thus, the number of distinct lines with $\kappa = 3$ and given A, B is

$$\begin{aligned} &\ll \# \{ (a, b) : a \ll D, b \ll E, ab^3 \ll U, a^3b \ll V \} \\ &\ll \# \{ (a, b) : ab \ll (UV)^{1/4} \} \ll (UV)^{1/4+\delta}. \end{aligned}$$

The number of points on each line is

$$\ll \min \left\{ \frac{D}{aA^3B^2}, \frac{E}{bA^2B^3} \right\} \ll \frac{\min(D, E)}{A^2B^2}.$$

Thus, the total number of points on lines with $\kappa = 3$ counted by $\mathcal{N}(D, E)$ is

$$\sum_{A, B} (UV)^{1/4+\delta} \frac{\min(D, E)}{A^2B^2} \ll \min(D, E)(UV)^{1/4+\delta} \ll x^{1/2+\delta}.$$

Let

$$\mathcal{N}_0(D, E) = \{(d, e, u, v) \in \mathcal{N}(D, E) : (d, e, u, v) \text{ does not lie on a line contained in (1)}\}.$$

Above, we have just shown that

$$\mathcal{N}_0(D, E) \ll_{\delta} x^{\delta} M \ll_{\delta} x^{2\delta+1-2\psi/3}.$$

Secondly, as before we have

$$\mathcal{N}_0(D, E) \leq \mathcal{N}(D, E) \ll_{\delta} x^{3\delta+\min(\psi, 2-2\psi)/2+\max(1/4, 9\psi(1-\psi)/8)},$$

where essentially $1/2 \leq \psi \leq 3/4$. One can check that the worst value for the exponent occurs if $\psi \leq 2/3$ and then that the worst value for ψ must satisfy

$$1 - \frac{2}{3}\psi = \frac{9}{8}\psi(1 - \psi) + \frac{\psi}{2}.$$

Hence the critical value for ψ is $(55 - \sqrt{433})/54 = 0.6331\dots$ and we may deduce that $\mathcal{N}_0(D, E) \ll_{\delta} x^{3\delta+\omega}$ where $\omega = (26 + \sqrt{433})/81 \leq 0.5779 \leq 7/12 = 0.5833\dots$. By the above argument, the points counted by $\mathcal{N}(D, E)$ but not being elements of $\mathcal{N}_0(D, E)$ contribute $O(x^{1/2+\delta})$. This finishes the proof of Theorem 1.

3 The Proof of Theorem 2

We are now turning to the proof of Theorem 2. We define

$$\xi(n) = \prod_{i=1}^r \prod_{p^k | l_i(n)} p,$$

and for $w > 1$ let

$$\mathcal{P}(w) = \prod_{p < w} p$$

be the product of primes $p < w$. Furthermore, for $z > 1$ define

$$\mathcal{S}(z) = \sum_{\substack{x < n \leq 2x \\ (\xi(n), \mathcal{P}(z))=1}} 1.$$

That is, $\mathcal{S}(z)$ is the number of integers n in the interval $(n, 2n]$ so that $l_1(n), \dots, l_r(n)$ all do not have any k -th power prime divisor p^k with $p < z$. In particular, $N(2x) - N(x) = \mathcal{S}(O(x^{1/k}))$. We also define

$$\mathcal{S}_d(w) = \sum_{\substack{x < n \leq 2x \\ \xi(n) \equiv 0 \pmod{d} \\ (\xi(n), \mathcal{P}(w)) = 1}} 1.$$

Next, we will employ the following identity which is essentially Buchstab's identity. That is, for $1 < w < z$ observe that

$$\mathcal{S}(z) = \mathcal{S}(w) - \sum_{w \leq p < z} \mathcal{S}_p(p). \quad (20)$$

Applying the identity twice, we obtain for $w > 1$:

$$N(2x) - N(x) = \mathcal{S}(O(x^{1/k})) = \mathcal{S}(w) - \sum_{w \leq p \ll x^{1/k}} \mathcal{S}_p(w) + \sum_{w \leq q < p \ll x^{1/k}} \mathcal{S}_{pq}(q).$$

First, we will estimate the sum $\sum_p \mathcal{S}_p(w)$. We split the sum over p in two parts, the first range will be $w \leq p < y$ and for the second range $y \leq p \ll x^{1/k}$ a trivial estimate will suffice. Similarly to section 2.1 we can deduce that:

$$N(x; d) = \# \{x < n \leq 2x : \xi(n) \equiv 0 \pmod{d}\} = \rho(d) \left(\frac{x}{d^k} + O(1) \right). \quad (21)$$

Thus,

$$\begin{aligned} \sum_{y \leq p \ll x^{1/k}} \mathcal{S}_p(w) &\ll \sum_{y \leq p \ll x^{1/k}} N(x; p) \ll x^\epsilon \sum_{y \leq p \ll x^{1/k}} \left(\frac{x}{p^k} + 1 \right) \\ &\ll x^\epsilon (xy^{1-k} + x^{1/k}) \ll x^{1+\epsilon} y^{1-k}. \end{aligned}$$

To estimate the the sum over the remaining range $w \leq p < y$ we will apply the following Fundamental Sieve Lemma due to Heath-Brown [7] which adapted to our purpose states as follows:

Lemma 6. *For $z > 1$ and $w > 1$ we have that*

$$\sum_{d | (\xi(n), \mathcal{P}(w))} \mu(d) = \sum_{\substack{d | (\xi(n), \mathcal{P}(w)) \\ d < z}} \mu(d) + O \left(\sum_{\substack{d | (\xi(n), \mathcal{P}(w)) \\ z \leq d < zw}} 1 \right).$$

Observe that a prime $p > w > 1$ and some $d \mid \mathcal{P}(w)$ both divide $\xi(n)$ if and only if pd divides $\xi(n)$. Thus, we pick some parameter $z_p > 1$ which we will determine later and

we split $\mathcal{S}_p(w)$ using the Lemma as follows:

$$\begin{aligned}
\mathcal{S}_p(w) &= \sum_{\substack{x < n \leq 2x \\ \xi(n) \equiv 0 \pmod{p}}} \sum_{d | (\xi(n), P(w))} \mu(d) \\
&= \sum_{\substack{x < n \leq 2x \\ \xi(n) \equiv 0 \pmod{p}}} \sum_{\substack{d | (\xi(n), \mathcal{P}(w)) \\ d < z_p}} \mu(d) + O \left(\sum_{\substack{x < n \leq 2x \\ \xi(n) \equiv 0 \pmod{p}}} \sum_{\substack{d | (\xi(n), \mathcal{P}(w)) \\ z_p \leq d < z_p w}} 1 \right) \\
&= \sum_{\substack{d | \mathcal{P}(w) \\ d < z_p}} \mu(d) N(x; pd) + O \left(\sum_{\substack{d | \mathcal{P}(w) \\ z_p \leq d < z_p w}} N(x; pd) \right) \\
&= S_1(p) + O(S_2(p)),
\end{aligned}$$

say. For $S_2(p)$ we obtain

$$S_2(p) \ll x^\epsilon \sum_{z_p \leq d < z_p w} \left(\frac{x}{d^k p^k} + 1 \right) \ll x^\epsilon \left(\frac{x}{p^k z_p^{k-1}} + z_p w \right).$$

We will pick $z_p = p^{-1}(x/w)^{1/k}$ to minimize this error term. Here we set $y = pz_p$ to ensure that $z_p > 1$. Using the fact that $\sum_{p \leq x} p^{-k} \ll x^\epsilon$ we can deduce that

$$\sum_{w \leq p < y} S_2(p) \ll x^{1/k+\epsilon} w^{1-1/k}.$$

Next, we use the trivial estimate (21) again to conclude that

$$- \sum_{w \leq p < y} S_1(p) = x \sum_{w \leq p < y} \sum_{\substack{d | \mathcal{P}(w) \\ pd < y}} \frac{\mu(pd) \rho(pd)}{(pd)^k} + O(yx^\epsilon).$$

The double sum equals

$$x \sum'_{d < y} \frac{\mu(d) \rho(d)}{d^k} = x \sum'_{d=1}^{\infty} \frac{\mu(d) \rho(d)}{d^k} + O(x^{1+\epsilon} y^{1-k}) \quad (22)$$

where the \sum' restricts the sum to those $d \mid \mathcal{P}(x^{1/k})$ with exactly one exceptional prime divisor $p \mid d$ such that $p > w$. Thus, we have shown that

$$- \sum_{w \leq p \ll x^{1/k}} \mathcal{S}_p(w) = c^{(1)} x + O(x^{1/k+\epsilon} w^{1-1/k}),$$

where $c^{(1)}$ is the constant from (22), the sum over those d having exactly one large prime divisor $p > w$. Next, we consider the sum $\mathcal{S}(w)$. Similarly to the above argument, we may apply Lemma 6 for some $z > 1$ to obtain

$$\mathcal{S}(w) = \sum_{\substack{d|\mathcal{P}(w) \\ d < z}} \mu(d)N(x;d) + O\left(\sum_{\substack{d|P(w) \\ z \leq d < zw}} N(x;d)\right). \quad (23)$$

Again, we use a trivial estimate for the second sum which yields an error term $O(x^{1/k+\epsilon}w^{1-1/k})$ provided we chose $z = (x/w)^{1/k}$ optimally. The first sum in (23) is

$$x \sum_{\substack{d|\mathcal{P}(w) \\ d < z}} \frac{\mu(d)\rho(d)}{d^k} + O(z).$$

Thus, we have shown that

$$\mathcal{S}(w) = c^{(0)}x + O(x^{1/k+\epsilon}w^{1-1/k}),$$

where

$$c^{(0)} = \sum_{d|\mathcal{P}(w)} \frac{\mu(d)\rho(d)}{d^k}$$

is the sum over those d having no large prime divisors $p > w$. Recall that the overall main term is $cx = x \sum_{d=1}^{\infty} \mu(d)\rho(d)d^{-k}$. The sum $c - c^{(0)} - c^{(1)}$ is the sum over those d having at least 2 distinct prime divisors $> w$ and thus we have

$$c^{(0)} + c^{(1)} = c + O\left(x^{\epsilon} \sum_{d > w^2} \frac{1}{d^k}\right) = c + O(x^{\epsilon}w^{2-2k}).$$

We minimize the error terms by choosing $w = x^{1/(2k+1)}$ so that both our error terms $O(x^{1+\epsilon}w^{2-2k})$ and $O(x^{1/k+\epsilon}w^{1-1/k})$ become $O(x^{3/(2k+1)+\epsilon})$. Thus, we have shown

$$\mathcal{S}(w) - \sum_{w \leq p \ll x^{1/k}} \mathcal{S}_p(w) = cx + O(x^{3/(2k+1)+\epsilon}),$$

and it remains to find a bound for the sum $\sum_{w \leq q < p \ll x^{1/k}} \mathcal{S}_{pq}(q)$. First we consider the terms corresponding to those prime pairs $q < p$ with $pq \ll x^{1/k}$. A trivial estimate suffices to yield the bound

$$\sum_{\substack{w \leq q < p \ll x^{1/k} \\ pq \ll x^{1/k}}} \mathcal{S}_{pq}(q) \ll \sum_{\substack{w \leq q < p \ll x^{1/k} \\ pq \ll x^{1/k}}} N(x;pq) \ll x^{\epsilon} \sum_{\substack{w \leq q < p \ll x^{1/k} \\ pq \ll x^{1/k}}} \left(\frac{x}{p^k q^k} + 1\right) \ll x^{3/(2k+1)+\epsilon}.$$

For the values with $pq \gg x^{1/k}$ observe that

$$\mathcal{S}_{pq}(q) \ll N(x;pq) = \#\left\{x < n \leq 2x : p^k \mid l_i(n), q^k \mid l_j(n) \text{ for some } i \neq j\right\} k$$

(The case $i = j$ cannot occur since $pq \gg x^{1/k}$ for any suitable implied constant). Thus, we can fix a particular i and j and conclude that

$$\sum_{\substack{w \leq q < p \ll x^{1/k} \\ pq \gg x^{1/k}}} \mathcal{S}_{pq}(q) \ll_r \#\mathcal{K},$$

where

$$\mathcal{K} = \{(p, q, u, v) : w \leq q < p \ll x^{1/k}, pq \gg x^{1/k}, p^k u = l_i(n), q^k v = l_j(n)\}.$$

Without loss of generality we may write $l_i(n) = a_1 n + b_1$ and $l_j(n) = a_2 n + b_2$ so that we are left with the Diophantine equation $a_1 q^k v - a_2 p^k u = a_1 b_2 - a_2 b_1 = h \neq 0$ say. Note that $(a_1 q^k v, a_2 p^k u) = (a_1 v, a_2 u)$ since neither p nor q divide h since h is $O(1)$ and p and q are $\gg x^{1/(2k+1)}$. Each common divisor of $a_1 v$ and $a_2 u$ is a divisor of h and thus is $O(1)$. Since also a_1 and a_2 are $O(1)$ we can reduce our problem to $d(h) = O(1)$ equations of the form (1) with the additional constraint that $(d^k u, e^k v) = 1$ and $de \gg x^{1/k}$. Thus, we have reduced the problem to the case we dealt with in Theorem 1. Thus, this gives an error term $O(x^{\max\{\omega(k), 3/(2k+1)\} + \epsilon})$ for the asymptotic formula in Theorem 2. Note that $\omega(k) \leq 3/(2k+1)$ for $k = 2, 3$ which concludes the proof.

4 The Proof of Theorem 3

We are now turning to the problem of consecutive square-full integers. Recall that here, $N(x)$ is the number of integers $n \leq x$ such that both n and $n+1$ are square-full. Observe that every square-full integer n can uniquely be written as $n = a^2 b^3$ with $\mu^2(b) = 1$. Thus, we have

$$N(2x) - N(x) \ll \#\{(d, e, u, v) \in \mathbb{N}^4 : x < d^3 u^2 = e^3 v^2 - 1 \leq 2x\}.$$

As in the proof of Theorem 1 we can now split the ranges of d and e into $O(x^\epsilon)$ boxes with $D/2 < d < D$ and $E/2 < e \leq E$ where $D, E \ll x^{1/3}$. For one such box we then get

$$N(2x) - N(x) \ll x^\epsilon \mathcal{N}(D, E),$$

where

$$\mathcal{N}(D, E) = \#\{(d, e, u, v) \in \mathbb{N}^4 : x < d^3 u^2 = e^3 v^2 - 1 \leq 2x, d \asymp D, e \asymp E\}.$$

Thus, in order to prove Theorem 3, it remains to show that

$$\mathcal{N}(D, E) \ll_\epsilon x^{29/100 + \epsilon}.$$

Our overall strategy is to apply the Determinant method similarly to the proof of Theorem 1. We define $y := x^{29/100}$ and $U := x^{1/2} D^{-3/2}$ and $V := x^{1/2} E^{-3/2}$ so that $v \asymp V$ and $u \asymp U$. Next observe that for fixed u, v :

$$\#\{(d, e) \in \mathbb{N}^2 : x < d^3 u^2 = e^3 v^2 - 1 \leq 2x, d \asymp D, e \asymp E\} \ll 1. \quad (24)$$

This is because the equation $e^3v^2 - d^3u^2 = 1$ is a Thue-equation in d and e and as remarked before the number of integer solutions of such an equation is $O(1)$. We also observe by Estermann's result that for fixed d, e :

$$\# \{(u, v) \in \mathbb{N}^4 : x < d^3u^2 = e^3v^2 - 1 \leq 2x, d \asymp D, e \asymp E\} \ll x^\epsilon. \quad (25)$$

These observations thus lead to the trivial estimate

$$\mathcal{N}(D, E) \ll x^\epsilon \min(DE, UV).$$

In particular, we may assume that $DE \geq y$. We now want to show as in the proof of Theorem 1 that $\max(U/V, V/U) < M$ where we will later pick $M \geq x^{9/50}$. Thus, it will suffice to show that $\max(U/V, V/U) < x^{17/100}$. There will be two cases to consider. First assume that $DE^{-3} \leq 1$. Then, similarly to (2) we get a trivial bound

$$\mathcal{N}(D, E) \ll EU \left(\frac{D}{E^3} + 1 \right) x^\epsilon \ll EU x^\epsilon,$$

and thus, we may assume that $EU \geq y$. Note that by definition of U ,

$$y^5 \leq (DE)(EU)^4 = \left(\frac{E}{D} \right)^5 x^2,$$

so that

$$\frac{V}{U} = \left(\frac{D}{E} \right)^{3/2} \leq \frac{x^{3/5}}{y^{3/2}} = x^{33/200} < x^{17/100}.$$

Next, we consider the case $DE^{-3} \geq 1$. We now fix a particular pair e, u and consider the solutions (d, e, u, v) counted by $\mathcal{N}(D, E)$. Let d_0 be the residue class of d modulo e^3 and let v_0 be the residue class of v modulo u^2 . Define integers a, b by the equations

$$v = v_0 + au^2, \text{ and } d = d_0 + be^3. \quad (26)$$

Note that each pair (d, v) determines (a, b) up to $O(1)$ choices because $DE^{-3} \geq 1$. Thus, an upper bound for the number of pairs (a, b) will give an upper bound for the number of pairs (u, v) . Substituting (26) into the equation $e^3v^2 - d^3u^2 = 1$ gives

$$p(a) - q(b) - 1 = 0, \quad (27)$$

where p is a polynomial of degree 2 and q is a polynomial of degree 3.

We will now prove that the left-hand side of (27) is absolutely irreducible. It is enough to show that the polynomial $f(S, T) = S^2 - T^3 - 1$ is absolutely irreducible. Assume that $f(S, T) = g(S, T)h(S, T)$ in some finite extension of \mathbb{Q} . In particular, $X^6 - Y^6 - 1 = g(X^3, Y^2)h(X^3, Y^2)$. We may homogenize the equation to get

$$X^6 - Y^6 - Z^6 = G(X, Y, Z)H(X, Y, Z) \quad (28)$$

for some polynomials G, H . Now let (X, Y, Z) be a nonzero point such that $G(X, Y, Z) = H(X, Y, Z) = 0$. The gradient of the right-hand side of (28) vanishes whereas the

gradient of the left-hand side is $(6X^5, -6Y^5, -6Z^5)$ which implies that $X = Y = Z = 0$. Contradiction. Thus, the equation (27) must be absolutely irreducible.

Next, we will proceed to apply Theorem 15 of Heath-Brown [5] to get an upper bound for the number of pairs (a, b) satisfying (27). Using the notation of Heath-Brown's theorem we will set $n := 2$, $B_1 \asymp \max(VU^{-2}, 1)$ and $B_2 \asymp DE^{-3}$ so that indeed $a \leq B_1$ and $b \leq B_2$ with $B_1, B_2 \geq 1$. Note that $T = \max(B_1^2, B_2^3) \geq B_1^2$ so that the points (a, b) satisfying (27) lie on at most $k \ll_\epsilon x^\epsilon B_2^{1/2}$ auxiliary curves. Thus, using Bézout's Theorem, we may deduce that the number of points (a, b) under consideration is $\ll_\epsilon x^\epsilon B_2^{1/2}$. Thus, we get the estimate

$$\mathcal{N}(D, E) \ll_\epsilon x^\epsilon EU \frac{D^{1/2}}{E^{3/2}} = \frac{x^{1/2+\epsilon}}{DE^{1/2}}.$$

Hence, we may assume that $x^{1/2+\epsilon} D^{-1} E^{-1/2} \geq y$. Thus,

$$y^7 \leq \left(\frac{x^{1/2}}{DE^{1/2}} \right)^4 (DE)^3 = x^2 \frac{E}{D},$$

so that

$$\frac{D}{E} \leq \frac{x^2}{y^7} \leq 1.$$

Thus, we have shown that in all cases, $V/U < x^{17/100}$. By interchanging the roles of D and E , we may similarly prove that $U/V < x^{17/100}$ and hence we conclude that

$$\max(D/E, E/D, U/V, V/U) < x^{17/100}. \quad (29)$$

We now consider the equation $e^3 v^2 - d^3 u^2 = 1$ and write it as

$$t^2 - s^3 = O(U^{-2} E^{-3}),$$

where $t = v/u$ and $s = d/e$. Note that $t^2 - s^3 = (t - s^{3/2})(t + s^{3/2})$ where $t + s^{3/2}$ is of exact order V/U so that we can conclude

$$t = s^{3/2} + O\left(\frac{1}{x} \frac{V}{U}\right).$$

We will now sketch the remaining parts of the proof. First, we proceed as in section 2.2 and we split the range of s into $O(M)$ intervals of the form $I = (s_0, s_0 + \frac{1}{M} \frac{D}{E}]$ where $M \in [x^{9/50}, x]$. We will then consider the solutions (s, t) with $s \in I$ where $s = s_0(1 + \alpha)$ and $t = s_0^{3/2}(1 + p(\alpha) + \beta)$. Here p is a polynomial of degree 5 (note that $M^{-6} \leq x^{-1}$) and $|\alpha_j| \leq X_1^{-1}$ and $|\beta_j| \leq X_2^{-1}$ where X_1 is of exact order M and X_2 is of exact order x as before. We then consider the determinant corresponding to Δ in the proof of Theorem 1. We can see that now

$$\Delta = s_0^{\frac{1}{2}H(K+3L/2)} \Delta_1,$$

where Δ_1 is again a generalized Vandermonde Determinant containing polynomial entries in α and β . Exactly as in section 2.2 we will then obtain a lemma similar to Lemma 4 which will produce an absolutely irreducible bi-homogeneous auxiliary equation $F(d, e; u, v) = 0$ for each interval I provided that $M \in [x^{9/50}, x]$ satisfies

$$\log M \geq \frac{9}{8}(1 + \delta) \frac{\log(DE) \log(UV)}{\log x}.$$

The counting argument in section 2.3 and section 2.4 then goes through as before. We have that:

$$s = s_0 + O\left(\frac{1}{M} \frac{D}{E}\right), \text{ where } s_0 = \frac{x_3}{M} \frac{D}{E} \text{ and } x_3 \asymp M, \quad \text{and}$$

$$t = t_0 + O\left(\frac{1}{M} \frac{V}{U}\right), \text{ where } t_0 = \frac{x_4}{M} \frac{V}{U} \text{ and } x_4 = \left\lfloor x_3^{3/2} M^{-1/2} \right\rfloor \asymp M.$$

The proof of the analogue of Lemma 5 will then use (24) and (25). The Thue equation under consideration will be $G_3(\tau_1, \tau_2) = q^3$ which will not change anything in the argument. The argument corresponding to section 2.4 will then use (29) and lead to the bound

$$\frac{\log \mathcal{N}(D, E)}{\log x} \leq 3\delta + \frac{1}{2} \min(\psi, 1 - 3\psi/2) + \frac{1 + \delta}{2} \max\left\{\frac{9}{8}\psi(1 - 3\psi/2), \frac{9}{50}\right\}.$$

Note that

$$\frac{1}{2} \min(\psi, 1 - 3\psi/2) + \max\left\{\frac{9}{16}\psi(1 - 3\psi/2), \frac{9}{100}\right\} \leq \frac{29}{100},$$

where the worst value is achieved for $\psi = 2/5$. This completes the proof of Theorem 3.

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